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Spin-$\frac{1}{2}$ Equation with Indefinite Metric*

RALPH F. GUERTIN

Physics Department, Rice University, Houston, Texas 77001

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Abstract

We study a Schrödinger equation involving a Hamiltonian that is a second-order differential operator, describes free spin-$\frac{1}{2}$ particles with both energy signs and a definite mass, and depends on a parameter G. One obtains the usual Dirac Hamiltonian by setting $G = \pm i$, but for real values of G the one-particle theory developed here possesses an indefinite metric, so negative energy states have negative normalization. Although the new equation is not manifestly covariant, it is demonstrated that it can be made invariant under proper orthochronous Poincaré transformations; it is also invariant under the CPT transformation and charge conjugation, but not, as we interpret it, under space inversion.

1. *Introduction*

Foldy (1956) has speculated on the existence of covariant wave equations whose second quantization requires the opposite of the usual spin-statistics relation; to illustrate this possibility we consider here and in another paper (Guertin, 1975b) that is to follow a new equation describing free spin-$\frac{1}{2}$ particles. In this paper we discuss the indefinite metric associated with our equation when it is interpreted as a one-particle equation, its discrete symmetry properties, and its invariance under proper orthochronous Poincaré transformations. The next paper will consider the sense in which second quantization is consistent with Bose statistics.

Our equation involves a Hamiltonian, H, that is a four by four matrix depending on a parameter G. This Hamiltonian is also a second-order differential operator whose square yields the familiar relativistic relation between the squares of the energy, the momentum, and the mass, so it describes free particles with both energy signs and a definite mass. If $G = \pm i$, it is equal to the familiar Dirac Hamiltonian that describes spin-$\frac{1}{2}$ particles, and the theory possesses a positive definite metric. On the other hand, for real values of $G \neq 0$, H also describes spin-$\frac{1}{2}$ particles but is pseudo-Hermitian; i.e.,

$$H^{\dagger} = \rho_3 H \rho_3$$

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and the indefinite metric leads to a negative normalization for negative energy states, as is the case in the Sakata–Taketani spin-0 and spin-1 theories (Sakata & Taketani, 1940; see also Heitler, 1943) and their generalizations to arbitrary spin (Guertin, 1974, 1975a). In contrast to the Dirac equation, which is invariant separately under space inversion, time inversion, and charge conjugation, the new equation, although invariant under CPT and under charge conjugation, is not separately invariant under either space inversion or time inversion. We are careful to explain in Section 3 why the metric is required to be indefinite, since one can always force either a positive definite metric or an indefinite one on a free-particle Hamiltonian, and also to explain in Section 4 why we do not interpret the theory as being invariant under space inversion.

Although the new equation cannot be manifestly covariant, it can be made invariant under proper orthochronous Poincaré transformations, as we discuss in Sections 5 and 6. If Λ is any transformation belonging to the proper orthochronous Lorentz group, then the wave function in a manifestly covariant theory such as that associated with the Dirac equation has the transformation property

$$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x)$$

where S is some matrix that acts only on the discrete indices of ψ. In the new theory, however, S also depends on the differential operator ∇, except when Λ is a rotation about some space axis. In the next paper, p. 405, we will demonstrate that when the theory is second-quantized the field and its adjoint *commute* for spacelike distances. The reason why this does not contradict the usual spin-statistics proofs (e.g., Pauli, 1940; Streater and Wightman, 1964) is that manifest covariance has been one of the assumptions upon which such proofs have been based.

The treatment in this and the following paper should be regarded only as the starting point in the construction of a theory describing spin-$\frac{1}{2}$ bosons, and it may happen that there is no consistent way in which to introduce interactions, a matter that is discussed in Section 8. However, if there is such a way, it will be extremely interesting to learn the physical significance of the real parameter G that appears in the new equation.

2. *The Hamiltonian*

We wish to consider the possibility of describing free massive particles by using as the Hamiltonian the four by four matrix operator

$$H = (\rho_3 + i\rho_2)(1 + G^2)p^2/2m + iG\rho_1 \boldsymbol{\sigma}\cdot\mathbf{p} + \rho_3 m \tag{2.1}$$

where the ρ matrices are formally equivalent to the Pauli matrices but commute with them. Here $p = |\mathbf{p}|$ and $m > 0$ is real. Regardless of the momentum dependence of G one has

$$H^2 = E^2 \tag{2.2a}$$

where

$$E = (p^2 + m^2)^{1/2} \tag{2.2b}$$

However, in this paper we shall omit a possible momentum dependence of G and assume that it is a constant. Thus, H is of second order in $\mathbf{p} = -i\nabla$ and is a local operator. For the present G can be any complex number, but in the next section we shall present arguments to restrict it to certain values.

The coordinate space wave function, $\psi(\mathbf{x}, t)$, is assumed to satisfy the Schrödinger equation

$$i\, \partial\psi/\partial t = H\psi \tag{2.3}$$

with $\mathbf{p} = -i\nabla$. Because of (2.2), ψ also satisfies the Klein-Gordon equation

$$(\partial^2/\partial t^2 - \nabla^2 + m^2)\psi = 0 \tag{2.4}$$

In fact, for $G = \pm i$, (2.3) is the Dirac equation, and for $G = 0$ it has the appearance of the Sakata-Taketani spin-0 equation (Sakata & Taketani, 1940; Heitler, 1943), although it has twice as many components as the latter.

To emphasize that the Hamiltonian (2.1) describes particles of mass $\pm m$ having both energy signs, we note that

$$\begin{aligned} \dot{\mathbf{x}} &= i[H, \mathbf{x}]_- \\ &= (\rho_3 + i\rho_2)(1 + G^2)\mathbf{p}/m + iG\rho_1\boldsymbol{\sigma} \end{aligned} \tag{2.5}$$

is such that

$$\dot{\mathbf{x}} \neq \mathbf{p}H^{-1} = \mathbf{p}HE^{-2} \tag{2.6}$$

Thus (Guertin and Guth, 1973; see also Guth, 1962), the matrix elements of $\mathbf{x}$ exhibit Zitterbewegung in the "charge space" of positive and negative energy states, and the presence of both signs of the energy requires the dimension of the Hamiltonian to be twice that of the number of spin degrees of freedom. Since H has been assumed to be a four by four operator, it must describe either spin-$\frac{1}{2}$ particles or spin-0 particles.

Let us stress that the operator $\mathbf{p}$, whose components satisfy

$$[p_i, p_j]_- = 0 \tag{2.7a}$$

is the canonical momentum operator and generates space translations. The relations

$$[p_i, H]_- = 0 \tag{2.7b}$$

express the invariance of the Hamiltonian under such translations. The invariance of (2.3) under rotations is assured by demonstrating the existence of an angular momentum operator $\mathbf{J}$ satisfying

$$[J_i, p_j]_- = i \sum_k \epsilon_{ijk} p_k \tag{2.7c}$$

$$[J_i, H]_- = 0 \tag{2.7d}$$

$$[J_i, J_j]_- = i \sum_k \epsilon_{ijk} J_k \tag{2.7e}$$

and by requiring that under an infinitesimal rotation of magnitude $|\boldsymbol{\theta}|$ about the direction $\hat{\theta} = \boldsymbol{\theta}/|\boldsymbol{\theta}|$ the wave function transforms into

$$\psi'(\mathbf{x}, t) = [1 - i\boldsymbol{\theta}\cdot\mathbf{J}]\,\psi(\mathbf{x}, t) \tag{2.8}$$

The spin is determined by $\mathbf{J}$ in the rest frame, but equations (2.8) alone are not sufficient to uniquely fix $\mathbf{J}$ or to determine the spin; i.e., it is possible to show that (2.7) allows either spin-$\frac{1}{2}$ or spin-0. In the former case there is a single irreducible representation for each energy sign, whereas in the latter case there are two such representations.

In order to guarantee that the Hamiltonian (2.1) describes spin-$\frac{1}{2}$ particles, an additional assumption is needed. One may, for example, require that it be possible to decompose $\mathbf{J}$ into an orbital part, $\mathbf{x} \times \mathbf{p}$, and a spin part, $\mathbf{S}$, that commutes with $\mathbf{x}$; this requirement is equivalent to assuming that $\mathbf{x}$ transforms like a three-vector under rotations; i.e., that

$$[J_i, x_j]_- = i \sum_k \epsilon_{ijk} x_k \tag{2.9}$$

One finds that when G does not identically vanish, equations (2.7) and (2.9) require that

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} + \boldsymbol{\sigma}/2 \tag{2.10}$$

and spin $\frac{1}{2}$ is the unique possibility, but that when $G = 0$ one may have either (2.10) or

$$\mathbf{J} = \mathbf{x} \times \mathbf{p}$$

which allows spin-0 particles. To restrict our considerations to spin-$\frac{1}{2}$ particles for *all* values of G, we may require that the spaces of positive and negative energy states be separately irreducible; for $G \neq 0$ this is equivalent to (2.9), but it is a stronger assumption than (2.9) when $G = 0$. Another possible assumption is that the case $G = 0$ be consistent with the limit of the more general case as $G \to 0$.

3. *The Metric*

The reality of the expectation value of any observable $\mathcal{O}$ requires the existence of a Hermitian metric operator M such that

$$M\mathcal{O} = \mathcal{O}^\dagger M \tag{3.1}$$

In particular, (3.1) must be valid when $\mathcal{O}$ is either H, $\mathbf{p}$, or $\mathbf{J}$, but since we are not requiring $\mathbf{x}$ to be an observable, it need not be consistent with (3.1). The normalization of the wave function is given by

$$\langle\psi, \psi\rangle_M = \int d^3x\,\psi^\dagger(\mathbf{x}, t) M \psi(\mathbf{x}, t) \tag{3.2}$$

and the expectation value of any observable $\mathcal{O}$ by

$$\langle\psi, \mathcal{O}\psi\rangle_M = \int d^3x\,\psi^\dagger(\mathbf{x}, t) M\mathcal{O}\psi(\mathbf{x}, t) \tag{3.3}$$

One can demonstrate that for a free particle theory there always exists a positive definite metric M_+ and an indefinite metric M_-; in the latter case $\langle\psi, \psi\rangle_{M_-}$ is positive if ψ contains only positive energies and negative if it contains only negative energies. For example, when $G = \pm i$ in (2.1), in which case one has the Dirac Hamiltonian, one may use $M_+ = I$ and $M_- = E^{-1}H$. When G is real one may use $M_+ = \rho_3 E^{-1}H$ and $M_- = \rho_3$, the same forms one has in the Sakata-Taketani spin-0 and spin-1 theories. But the use of an indefinite metric with the Dirac Hamiltonian or of a positive definite one with the Sakata-Taketani Hamiltonian leads to inconsistencies when one second quantizes the equations in the usual manner. In order that the energy be positive definite in the second-quantized theory, one must employ Bose statistics when the metric is indefinite and Fermi statistics when it is positive definite (e.g., Pauli, 1940). On the other hand, it is also generally assumed (e.g., Streater & Wightman, 1964) that causality requires either $[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t')]_-$ or $[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)]_+$ to vanish when the separation of the points $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$ is spacelike, an assumption that requires Fermi statistics for a Dirac field and Bose statistics for a Sakata-Taketani field. Thus, only the choice $M_+ = I$ leads to a consistent second-quantized Dirac theory, and only the choice $M_- = \rho_3$ leads to such a Sakata-Taketani theory.

Since we are not at the present time concerned with second quantization and wish to confine our discussion to the first-quantized theory based on the Hamiltonian (2.1), it is interesting to note that the acceptable metrics in the Dirac and Sakata-Taketani theories both satisfy

$$[M, \mathbf{x}]_- = 0 \tag{3.4}$$

and are thus independent of $\mathbf{p}$. This means that when interactions are introduced there is no difficulty in preserving the relation

$$MH = H^\dagger M \tag{3.5}$$

whereas if M depends on $\mathbf{p}$ in the free-particle case it is by no means obvious how to define M in the interacting case so that (3.5) is valid. In fact, although it was not so stated there, this was the author's main motivation for assuming the relations (3.4) for the relativistic Hamiltonian theories discussed elsewhere (Guertin, 1974, 1975).

For the reasons introduced in the preceding paragraph, let us therefore postulate that (3.4) must be valid for a theory based on the Hamiltonian (2.1). Then, either (i) $G = \pm i$ and $M = I$, in which case one has the Dirac Hamiltonian and observables are Hermitian, or (ii) G is real and $M = \rho_3$, in which case our equation (2.3) becomes a candidate for a new spin-$\frac{1}{2}$ theory that is not equivalent to that of Dirac because of the indefinite metric.

4. *Discrete Symmetries*

For a theory such as the one established so far to be invariant under space inversion there must exist a linear operator $\mathscr{S}$ such that (Foldy, 1956; Guertin, 1974)

$$[\mathscr{S}, \mathbf{p}]_+ = 0 \tag{4.1a}$$

$$[\mathscr{S}, \mathbf{J}]_- = 0 \tag{4.1b}$$

$$[\mathscr{S}, H]_- = 0 \tag{4.1c}$$

Then (2.3) is invariant under space inversion if the wave function ψ transforms into

$$\psi'(\mathbf{x}, t) = \mathscr{S}\psi(\mathbf{x}, t) \tag{4.2}$$

In order to have invariance under time inversion there must exist an antilinear operator $\mathscr{T}$ with the properties (Foldy, 1956; Guertin, 1974)

$$[\mathscr{T}, \mathbf{p}]_+ = 0 \tag{4.3a}$$

$$[\mathscr{T}, \mathbf{J}]_+ = 0 \tag{4.3b}$$

$$[\mathscr{T}, H]_- = 0 \tag{4.3c}$$

and, in addition, ψ must transform into

$$\psi'(\mathbf{x}, t) = \mathscr{T}\psi(\mathbf{x}, -t) \tag{4.4}$$

For the theory to be invariant under charge conjugation one must have an antilinear operator $\mathscr{C}$ such that (Foldy, 1956; Guertin, 1974)

$$[\mathscr{C}, \mathbf{p}]_+ = 0 \tag{4.5a}$$

$$[\mathscr{C}, \mathbf{J}]_+ = 0 \tag{4.5b}$$

$$[\mathscr{C}, H]_+ = 0 \tag{4.5c}$$

and, furthermore, ψ must transform into

$$\psi'(\mathbf{x}, t) = \mathscr{C}\psi(\mathbf{x}, t) \tag{4.6}$$

There is also the possibility of having a theory that is invariant under some combination of these operations, although not under the individual ones. Thus, invariance under the CPT transformation (simultaneous charge conjugation and space-time inversion) results from the existence of a linear operator $\mathscr{R}$ with the properties

$$[\mathscr{R}, \mathbf{p}]_+ = 0 \tag{4.7a}$$

$$[\mathscr{R}, \mathbf{J}]_- = 0 \tag{4.7b}$$

$$[\mathscr{R}, H]_+ = 0 \tag{4.7c}$$

provided that ψ is transformed into

$$\psi'(\mathbf{x}, t) = \mathscr{R}\psi(\mathbf{x}, -t) \tag{4.8}$$

The square of any discrete symmetry operator should have no physical effect. However, since quantum mechanics requires only a ray representation, one can only conclude that if $\mathscr{F}$ is a discrete symmetry operator for the physical system

$$\mathscr{F}^2 = \epsilon_F \tag{4.9a}$$

where $|\epsilon_F| = 1$. Furthermore, if $\mathscr{G}$ is any other symmetry operator for the system whose physical effect commutes with that of $\mathscr{F}$, then the "equality up to a phase factor η_{FG}"

$$\mathscr{F}\mathscr{G} = \eta_{FG}\mathscr{G}\mathscr{F} \tag{4.9b}$$

is all that quantum theory requires (see, e.g., Wightman, 1960; Wigner, 1964). If a physical theory of the type being discussed is invariant under space inversion, time inversion, and charge conjugation, one usually postulates that the corresponding operators are consistent with (4.9). Then (e.g., Guertin, 1974) one can always choose $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{C}$ such that

$$\mathscr{S}^2 = 1 \tag{4.10a}$$

$$\mathscr{T}^2 = (-1)^{2J} \tag{4.10b}$$

$$\mathscr{C}^2 = -\epsilon(-1)^{2J} \tag{4.10c}$$

$$\mathscr{S}\mathscr{T} = \mathscr{T}\mathscr{S} \tag{4.10d}$$

$$\mathscr{T}\mathscr{C} = \pm\mathscr{C}\mathscr{T} \tag{4.10e}$$

$$\mathscr{S}\mathscr{C} = \eta\mathscr{C}\mathscr{S} \tag{4.10f}$$

where $\epsilon = +1$ or -1 and where $\eta = +1$ or -1. There are thus four inequivalent possibilities, depending on the signs of ϵ and of η in (4.10c) and (4.10f), respectively [the two sign possibilities in (4.10e) are equivalent].

It was mentioned in Section 3 that one can always force either a positive definite metric or an indefinite metric on a free-particle theory of the type under discussion, but in general neither commutes with $\mathbf{x}$. Similarly, one can always find four inequivalent sets of operators $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{C}$ satisfying the properties (4.1), (4.3), and (4.5), each set being consistent with one of the four possibilities allowed by (4.10). However, in general none of these sets will be such that

$$[\mathscr{S}, \mathbf{x}]_+ = 0 \tag{4.11a}$$

$$[\mathscr{T}, \mathbf{x}]_- = 0 \tag{4.11b}$$

$$[\mathscr{C}, \mathbf{x}]_- = 0 \tag{4.11c}$$

the properties required if $\mathbf{x}$ is to transform like a vector. We earlier rejected a metric that did not commute with $\mathbf{x}$ because it is not clear how property (3.1) could continue to be satisfied when interactions are introduced. Similarly, it is difficult to see how the relations (4.1), (4.3), and (4.5) can be maintained in the presence of interactions if (4.11) is not valid. We shall therefore postulate (as in Guertin, 1974) that a necessary and sufficient condition for (2.3) to be invariant under space inversion is the existence of a space inversion operator $\mathscr{S}$ satisfying (4.1) and (4.11a). Likewise, we assume that (2.3) is invariant under time inversion if and only if there exists a time inversion operator $\mathscr{T}$ satisfying (4.3) and (4.11b). In addition, we state that a necessary and sufficient condition for (2.3) to be invariant under charge conjugation is the existence of a charge conjugation operator $\mathscr{C}$ with the properties (4.5) and (4.11c). Finally, we assume that (2.3) is invariant under CPT if and only if there exists a CPT operator $\mathscr{R}$ with the properties (4.7) and

$$[\mathscr{R}, \mathbf{x}]_+ = 0 \tag{4.12}$$

Let $\mathscr{P}$ be the familiar operator that anticommutes with $\mathbf{x}$ and with $\mathbf{p}$, but which commutes with $\boldsymbol{\sigma}$ and with the ρ matrices. Then

$$\mathscr{R} = \rho_1 \mathscr{P} \tag{4.13}$$

has all the required properties to be a CPT operator. It has been shown (Guertin, 1974) that a Hamiltonian theory with the metric $M = I$ or the metric $M = \rho_3$ always possesses a charge conjugation operator $\mathscr{C}$. Let κ be the operator that takes the complex conjugate of all expressions standing to the right of it and let

$$c = \exp(-i\pi\, \sigma_2/2) \tag{4.14}$$

Then

$$\mathscr{C}^{(1)} = \rho_2 c\kappa \tag{4.15a}$$

has the required properties if and only if $M = I$ and

$$\mathscr{C}^{(2)} = \rho_1 c\kappa \tag{4.15b}$$

has them if and only if $M = \rho_3$.

For the Dirac Hamiltonian, in which case $G = \pm i$ and $M = I$, the properties (4.3) and (4.11b) are satisfied by

$$\mathscr{T} = c\kappa \tag{4.16}$$

so $\mathscr{T}$ is a time inversion operator for the Dirac equation; one also finds that a space inversion operator consistent with (4.1) and (4.11a) is

$$\mathscr{S}^{(2)} = \pm\rho_3 \mathscr{P} \tag{4.17}$$

For real values of $G \neq 0$, no operators $\mathscr{S}$ and $\mathscr{T}$ that are consistent with (4.1), (4.3), and (4.11a) and (4.11b) exist, so the theory does not meet our criteria for invariance under either space or time inversion. However, from the invariance under both charge conjugation and the CPT transformation, we know that the theory is invariant under simultaneous space-time inversion. An appropriate operator is

$$\mathscr{L} = \mathscr{P} c\kappa \tag{4.18}$$

We also note that for $G = 0$, when $\boldsymbol{\sigma}$ does not appear in the Hamiltonian, the properties (4.1), (4.3), and (4.11a, b) are satisfied by $\mathscr{T}$ in (4.16) and

$$\mathscr{S}^{(1)} = \pm\mathscr{P} \tag{4.19}$$

One can rule out space inversion and time inversion invariance for the case $G = 0$ by requiring that it be consistent with the limit of the more general case as $G \to 0$ through real values.

One could rescue the space and time inversion invariance of the theory for real values of $G \neq 0$ by allowing a "parity doubling" to $4(2J + 1)$ components (for a discussion of how "parity doubling" can occur in a manifestly covariant theory, see, e.g., Hurley, 1971, 1974). Since such a procedure leads to other complications we shall not adopt it.

5. *Invariance under Proper Orthochronous Poincaré Transformations*

One can assure the invariance of the theory discussed so far under transformations belonging to the proper orthochronous Poincaré Group, $\mathscr{P}_+^\uparrow$, if there exists a boost operator $\mathbf{K}$ such that (Foldy, 1956; Guertin, 1974)

$$[p_i, K_j]_- = -i\delta_{ij}H \tag{5.1a}$$

$$[H, K_i]_- = -ip_i \tag{5.1b}$$

$$[J_i, K_j]_- = i\sum_k \epsilon_{ijk}K_k \tag{5.1c}$$

$$[K_i, K_j]_- = -i\sum_k \epsilon_{ijk}J_k \tag{5.1d}$$

and, furthermore,

$$\mathbf{K} = M\mathbf{K}^\dagger M \tag{5.2a}$$

$$[\mathscr{R}, \mathbf{K} + t\mathbf{p}]_- = 0 \tag{5.2b}$$

$$[\mathscr{C}, \mathbf{K}]_+ = 0 \tag{5.2c}$$

In addition, if operators $\mathscr{S}$ and $\mathscr{T}$ having the properties introduced in the preceding section exist, it is required that

$$[\mathscr{S}, \mathbf{K}]_+ = 0 \tag{5.3a}$$

$$[\mathscr{T}, \mathbf{K} + t\mathbf{p}]_- = 0 \tag{5.3b}$$

Then (2.3) is invariant under Lorentz boosts provided that under an infinitesimal boost determined by the real vector $\boldsymbol{\lambda}$ the wave function transforms into

$$\psi'(\mathbf{x}, t) = (1 + i\boldsymbol{\lambda}\cdot\mathbf{K})\psi(\mathbf{x}, t) \tag{5.4}$$

Note that the general form of the wave function $\psi(\mathbf{x}, t)$ is determined not only by the Hamiltonian but also by the particular generator chosen to generate boosts. If (5.1), (5.2), and also, when applicable, (5.3) do not uniquely determine $\mathbf{K}$, then one must adopt some additional criteria in order to specify a unique boost operator.

Instead of attempting to solve (5.1) for the most general form of **K**, one can demonstrate the possibility of basing a $\mathscr{P}_+^\uparrow$-invariant theory on the Hamiltonian (2.1) by demonstrating the equivalence of (2.3) to a Foldy canonical equation (Foldy, 1956)

$$i\,\partial\psi^F/\partial t = H_F\psi^F \tag{5.5}$$

where

$$H_F = \rho_3 E \tag{5.6}$$

and where ψ_F is the wave function in the Foldy canonical representation. This equivalence is established if there exists a generalized Foldy-Wouthuysen operator W (see, e.g., Foldy and Wouthuysen, 1950; Tani, 1951; Guertin, 1975a) such that

$$\mathbf{p} = W\mathbf{p}W^{-1} \tag{5.7a}$$

$$\mathbf{J} = W\mathbf{J}W^{-1} \tag{5.7b}$$

$$H = WH_FW^{-1} \tag{5.7c}$$

These relations do not uniquely determine W, but once one has given some criteria for doing so one obtains a $\mathscr{P}_+^\uparrow$-invariant theory by requiring that

$$\psi = W\psi^F \tag{5.8}$$

and then

$$\mathbf{K} = W\mathbf{K}_FW^{-1} \tag{5.9}$$

generates Lorentz boosts, where

$$\mathbf{K}_F = \tfrac{1}{2}[\mathbf{x}, H_F]_+ - \rho_3(E+m)^{-1}\,\tfrac{1}{2}\boldsymbol{\sigma}\times\mathbf{p} - t\mathbf{p} \tag{5.10}$$

is the boost generator in the Foldy canonical representation. In fact, corresponding to any other observable $\mathcal{O}$ in our representation there exists an operator $\mathcal{O}_F$ such that

$$\mathcal{O} = W\mathcal{O}_FW^{-1} \tag{5.11}$$

In Section 3 we restricted the metric to the two possibilities $M = I$ and $M = \rho_3$ by requiring it to commute with **x**. Since the normalization of the wave function and the expectation value of any observable are independent of the representation, one has

$$\langle\psi, \psi\rangle_M = \langle\psi^F, \psi^F\rangle_M \tag{5.12a}$$

$$\langle\psi, \mathcal{O}\psi\rangle_M = \langle\psi^F, \mathcal{O}_F\psi^F\rangle_M \tag{5.12b}$$

According to the above

$$W^{-1} = MW^\dagger M \tag{5.13}$$

so W is unitary (pseudounitary) if H is Hermitian (pseudo-Hermitian).

It is easy to write down the most general form for an operator W that satisfies (5.7) and (5.13). Let us first rewrite the Hamiltonian (2.1) using the helicity projection operators

$$\Lambda_{\hat{p}}^{\pm} = \tfrac{1}{2}(1 \pm \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}) \tag{5.14}$$

where $\hat{p} = \mathbf{p}/p$. These operators satisfy

$$\Lambda_{\hat{p}}^{\mu}\Lambda_{\hat{p}}^{\mu'} = \delta_{\mu\mu'}\Lambda_{\hat{p}}^{\mu} \tag{5.15}$$

Thus, one may write

$$H = \sum_{\mu} H_{\mu}\Lambda_{\hat{p}}^{\mu} \tag{5.16a}$$

where

$$H_{\pm} = (\rho_3 + i\rho_2)(1 + G^2)p^2/2m \pm iG\rho_1 p + \rho_3 m \tag{5.16b}$$

We may therefore write the Foldy-Wouthuysen operator and its inverse in the forms

$$W = \sum_{\mu} W_{\mu}\Lambda_{\hat{p}}^{\mu} \tag{5.17a}$$

$$W^{-1} = \sum_{\mu} W_{\mu}^{-1}\Lambda_{\hat{p}}^{\mu} \tag{5.17b}$$

Equations (5.7) and (5.13) are satisfied provided that

$$W_{\pm} = \Omega_{\pm}\exp[-i(\phi_{\pm} + \theta_{\pm}\rho_3)] \tag{5.18a}$$

$$W_{\pm}^{-1} = \exp[i(\phi_{\pm} + \theta_{\pm}\rho_3)]\,\Omega_{\pm}^{-1} \tag{5.18b}$$

where $\theta_{\pm}$ and $\phi_{\pm}$ are real functions of p and where

$$\Omega_{\pm} = \overline{\mathscr{G}}(E + H_{\pm}\rho_3) \tag{5.19a}$$

$$\Omega_{\pm}^{-1} = \overline{\mathscr{G}}(E + \rho_3 H_{\pm}) \tag{5.19b}$$

$$\overline{\mathscr{G}} = \left\{\frac{m}{E(E+m)[(1+G^2)E + (1-G^2)m]}\right\}^{1/2} \tag{5.19c}$$

The functions $\theta_{\pm}(p)$ and $\phi_{\pm}(p)$ must, of course, be consistent with (5.2b) and (5.2c) and, when applicable, (5.3), as will be discussed in the next section. One forces boost invariance on the theory by specifying the forms of these four functions and then defining the boost operator by (5.9) and the wave function by (5.8). Note that for the case $G = \pm i$ and $\theta_{\pm} = \phi_{\pm} = 0$ we have the familiar Foldy-Wouthuysen operator for the Dirac theory, and that for $G = \theta_{\pm} = \phi_{\pm} = 0$ the Foldy-Wouthuysen operator has the same form as the operator found by Case (1954) for the Sakata-Taketani Hamiltonian and generalized by the present author (Guertin, 1975a) to arbitrary spin.

6. *The Boost Generator*

The considerations of the preceding section were sufficient to demonstrate that a $\mathscr{P}_+^\uparrow$-invariant free-particle theory can be based on the Hamiltonian (2.1). In this section we shall discuss the boost generator **K**, in particular its nonlocal nature when G is real.

One can obtain the specific form of the boost generator from (5.9), (5.10), (5.14), and (5.17). The result is

$$\mathbf{K} = \tfrac{1}{2}[\mathbf{x}, H]_+ + \mathbf{\Gamma} - t\mathbf{p} \tag{6.1a}$$

where

$$\begin{aligned}\mathbf{\Gamma} = &-(1/4p^2) \sum_\mu [H_\mu - mW_\mu \rho_3 W_{-\mu}^{-1}]\,\boldsymbol{\sigma} \times \mathbf{p} \\ &-(im/4p^3) \sum_\mu (-1)^{(1-\mu)/2} W_\mu \rho_3 W_{-\mu}^{-1}(p^2 \boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\mathbf{p}\mathbf{p}) \\ &-(iE/4p^2) \sum_\mu [W'_\mu \rho_3 W_\mu^{-1} - W_\mu \rho_3 (W_\mu^{-1})'] \,[p + (-1)^{(1-\mu)/2}\boldsymbol{\sigma}\cdot\mathbf{p}]\,\mathbf{p}\end{aligned} \tag{6.1b}$$

Here we have adopted the convention that if F is any function of p, then F' is its derivative with respect to p. For $\eta_\mu = 1$ or $\eta_\mu = (-1)^{(1-\mu)/2}$ one has, according to (5.18),

$$\begin{aligned}&\sum_\mu \eta_\mu W_\mu \rho_3 W_{-\mu}^{-1} \\ &\quad = \sum_\mu \eta_\mu \exp[-i(\phi_\mu - \phi_{-\mu})]\,[\Omega_\mu \rho_3 \Omega_{-\mu}^{-1} \cos(\theta_+ - \theta_-) - i(-1)^{(1-\mu)/2}\Omega_\mu \Omega_{-\mu}^{-1} \\ &\qquad \sin(\theta_+ - \theta_-)]\end{aligned} \tag{6.2a}$$

Furthermore,

$$\begin{aligned}&W'_\pm \rho_3 W_\pm^{-1} - W_\pm \rho_3 (W_\pm^{-1})' \\ &\quad = \Omega'_\pm \rho_3 \Omega_\pm^{-1} - \Omega_\pm \rho_3 (\Omega_\pm^{-1})' - i2\theta'_\pm - i2\phi'_\pm H_\pm E^{-1}\end{aligned} \tag{6.2b}$$

One finds, using (5.19), that

$$\begin{aligned}\Omega_\pm \rho_3 \Omega_\mp^{-1} = &(\rho_3 + i\rho_2)(1 + G^2)\frac{p^2}{2mE} + \rho_3 \frac{m}{E}\left[\frac{(1 - G^2)E + (1 + G^2)m}{(1 + G^2)E + (1 - G^2)m}\right] \\ &\pm i\,\frac{G(1 + G^2)(E - m)p}{E[(1 + G^2)E + (1 - G^2)m]}\end{aligned} \tag{6.3a}$$

$$\Omega_\pm \Omega_\mp^{-1} = \mp i\left[\rho_3 \frac{(1+G^2)(E-m)}{(1+G^2)E+(1-G^2)m} + i\rho_2\right]\frac{Gp}{E} + \frac{(1+G^2)E(E+m)-2G^2m^2}{E[(1+G^2)E+(1-G^2)m]} \tag{6.3b}$$

$$\Omega'_\pm \rho_3 \Omega_\pm - \Omega_\pm \rho_3 (\Omega_\pm^{-1})' = \mp i \frac{(1+G^2)G}{E[(1+G^2)E+(1-G^2)m]} \tag{6.3c}$$

When the conditions (5.2b, c) are imposed on **K**, one finds that ϕ_+ and ϕ_- must be equal and independent of p and that, in addition,

$$\theta'_+ + \theta'_- = 0 \tag{6.4}$$

Then (6.2) and (6.3) simplify to yield, for the various terms in (6.1b),

$$\sum_\mu [H_\mu - mW_\mu \rho_3 W_{-\mu}^{-1}]$$
$$= (\rho_3 + i\rho_2)(1+G^2)\frac{p^2}{m} + 2\rho_3 m - \left\{(\rho_3 + i\rho_2)(1+G^2)\frac{p^2}{E} + \rho_3 \frac{2m^2}{E}\left[\frac{(1-G^2)E+(1+G^2)m}{(1+G^2)E+(1-G^2)m}\right]\right\}\cos(\theta_+ - \theta_-)$$
$$+ \frac{2Gpm}{E}\left[\rho_3 \frac{(1+G^2)(E-m)}{(1+G^2)E+(1-G^2)m} + i\rho_2\right]\sin(\theta_+ - \theta_-) \tag{6.5a}$$

$$\sum_\mu (-1)^{(1-\mu)/2} W_\mu \rho_3 W_{-\mu}^{-1}$$
$$= \frac{i2}{E[(1+G^2)E+(1-G^2)m]}\{[G(1+G^2)(E-m)p]\cos(\theta_+ - \theta_-)$$
$$- [(1+G^2)E(E+m) - 2G^2m^2]\sin(\theta_+ - \theta_-)\} \tag{6.5b}$$

$$\sum_\mu [W'_\mu \rho_3 W_\mu^{-1} - W_\mu \rho_3 (W_\mu^{-1})'] = 0 \tag{6.5c}$$

$$\sum_\mu (-1)^{(1-\mu)/2}[W'_\mu \rho_3 W_\mu^{-1} - W_\mu \rho_3 (W_\mu^{-1})'] = -i2\frac{(1+G^2)G(E-m)}{E[(1+G^2)E+(1-G^2)m]} - i2(\theta'_+ - \theta'_-) \tag{6.5d}$$

One may, furthermore, verify that if $G = \pm i$, then either of equations (5.3) requires θ_+ and θ_- to be equal and independent of p; one thus obtains $\mathbf{\Gamma} = 0$ in (6.1a), the usual result for the Dirac theory (Foldy, 1956; Fuschich *et al.* 1971; Kolsrud, 1971; Guertin, 1974, 1975a), and the theory is manifestly covariant. Thus, the manifest covariance of the Dirac theory is a consequence of its invariance under the various discrete symmetry operations.[1]

It is interesting to discuss further the conditions under which the wave function is manifestly covariant, i.e., those conditions under which the effects of an infinitesimal transformation belonging to $\mathscr{P}_+^\uparrow$ can always be expressed as the sum of an operator that acts only on the continuous space-time indices of ψ and of one that acts only on its four discrete indices that express the spin and "charge space" degrees of freedom. Under an infinitesimal space translation by the amount $\mathbf{d}$, the wave function transforms into

$$\psi'(\mathbf{x}, t) = [1 - \mathbf{d} \cdot \nabla]\, \psi(\mathbf{x}, t) \tag{6.6a}$$

and with the aid of (2.3) we see that the effect of an infinitesimal time translation by the amount D is to transform ψ into

$$\psi'(\mathbf{x}, t) = [1 - D\, \partial/\partial t]\, \psi(\mathbf{x}, t) \tag{6.6b}$$

In each of the above two examples only the space-time indices of ψ are affected by the transformation. On the other hand, for an infinitesimal rotation, (2.8) and (2.10) yield

$$\psi'(\mathbf{x}, t) = [1 - \boldsymbol{\theta} \cdot \mathbf{x} \times \nabla - i\boldsymbol{\theta} \cdot (\boldsymbol{\sigma}/2)]\, \psi(\mathbf{x}, t) \tag{6.7}$$

a relation that exhibits the decomposition of the transformation into a part affecting only the space-time indices and into a part affecting only the discrete indices. To consider boosts, we first note that (6.1a) may be written in the form

$$\mathbf{K} = \mathbf{x}H - t\mathbf{p} + \boldsymbol{\Gamma}' \tag{6.8a}$$

where

$$\boldsymbol{\Gamma}' = -\tfrac{1}{2} i\dot{\mathbf{x}} + \boldsymbol{\Gamma} \tag{6.8b}$$

with $\dot{\mathbf{x}}$ given by (2.5). From (2.3), (5.4), and the above we see that under an infinitesimal boost ψ goes into

$$\psi'(\mathbf{x}, t) = [1 - \boldsymbol{\lambda} \cdot (\mathbf{x}\, \partial/\partial t + t\nabla) + i\boldsymbol{\lambda} \cdot \boldsymbol{\Gamma}']\, \psi(\mathbf{x}, t) \tag{6.9}$$

Thus, ψ transforms in a manifestly covariant manner if $\boldsymbol{\Gamma}'$ depends only on the ρ matrices and on $\boldsymbol{\sigma}$. When $G = \pm i$, inspection of (6.1), (6.4), and (6.5) shows that necessary and sufficient conditions for manifest covariance are the equalities $\theta_+ = \theta_-$ and $\theta'_\pm = 0$; then $\boldsymbol{\Gamma}$ vanishes identically and

$$\boldsymbol{\Gamma}' = \mp i\rho_1\, \boldsymbol{\sigma}/2$$

There are no other values of G and $\theta_\pm$ that yield manifest covariance. Consequently, if one were to require the manifest covariance of the theory, as is usually done, only the Dirac equation would survive, and we shall therefore not impose such a requirement on our new equation.

[1] On the other hand, if one postulates manifest covariance instead of invariance under the various discrete symmetry operators, then operators $\mathscr{S}$, $\mathscr{C}$, and $\mathscr{T}$ consistent with (4.1), (4.3), (4.5), (4.11), (5.2), and (5.3) exist.

It is, of course, obvious from direct inspection of the Hamiltonian (2.1) and the Schrödinger equation (2.3) that the theory cannot be manifestly covariant for real values of G, since it is not symmetric in the space and time indices. We note that by introducing a six-component wave function Φ it is possible to reexpress (2.1) and (2.3) in the form

$$(\beta \cdot p - m)\Phi = 0 \tag{6.10}$$

and one might thus hope to obtain a manifestly covariant equation, just as one could obtain the five- and ten-component manifestly covariant Duffin-Kemmer-Petiau equations of the form (6.10) for spin 0 and spin 1 from the corresponding Sakata-Taketani Hamiltonians (for this relationship, see Sakata & Taketani, 1940; Heitler, 1943; Fischbach et al., 1972). However, this procedure does not lead to manifestly covariant equations for real values of G.

It is clear from (6.1b) and (6.5) that $\mathbf{\Gamma}$ is generally not a local operator for real values of G, i.e., not a simple polynomial in the components of $\mathbf{p}$. Can we, at least for one real value of G, find functions $\theta_\pm(p)$ and $\phi_\pm(p)$ such that $\mathbf{\Gamma}$, and therefore also $\mathbf{K}$, is a local operator? This would lead to criteria for uniquely fixing $\mathbf{K}$, but the author has not been able to find satisfactory functions $\theta_\pm(p)$ and $\phi_\pm(p)$ and does not believe it is possible to do so. Thus, if one wishes to construct a physical theory based on the Hamiltonian (2.1) with real values of G and the metric operator ρ_3, it appears that one must accept nonlocal transformation properties under boosts.

The present author has stressed elsewhere (Guertin, 1975a) that the generator of infinitesimal Poincaré transformations should be defined in the rest frame of a particle. The Hamiltonian (2.1) is certainly defined in the rest frame, but what can be said about the boost operator when G is real? Inspection of (6.1b), (6.4), and (6.5) shows that necessary and sufficient conditions for

$$\lim_{\mathbf{p}\to 0} \mathbf{K}$$

to exist are the equalities $\theta_+(0) = \theta_-(0)$ and $\theta'_+(0) = \theta'_-(0) = 0$.

7. *Continuity Equation*

Let us investigate the most general form of the continuity equation that one can obtain from the equation of motion (2.3) using the Hamiltonian (2.1). Let U and V be any nonsingular operators that satisfy

$$[U, H]_- = [V, H]_- = 0 \tag{7.1a}$$

$$V^\dagger MU + U^\dagger MV = 2M \tag{7.1b}$$

After first multiplying (2.3) from the left by $(V\psi)^\dagger MU + (U\psi)^\dagger MV$ and then subtracting the adjoint equation one finds that

$$\partial\rho/\partial t = -\nabla\cdot\mathbf{j} \tag{7.2}$$

where

$$\rho = \tfrac{1}{2}[(V\psi)^\dagger M(U\psi) + (U\psi)^\dagger M(V\psi)] \tag{7.3}$$

$$\begin{aligned}\mathbf{j} = -i[(1+G^2)/4m]\,&[(V\psi)^\dagger M(\rho_3 + i\rho_2)(\nabla U\psi) + (U\psi)^\dagger M(\rho_3 + i\rho_2)(\nabla V\psi)\\ &- (\nabla U\psi)^\dagger M(\rho_3 + i\rho_2)(V\psi) - (\nabla V\psi)^\dagger M(\rho_3 + i\rho_2)(U\psi)]\\ &+ i(G/2)[(V\psi)^\dagger M\rho_1\boldsymbol{\sigma}(U\psi) + (U\psi)^\dagger M\rho_1\boldsymbol{\sigma}(V\psi)]\end{aligned} \tag{7.4}$$

Furthermore, the "probability density" (7.2) is consistent with (3.2); i.e.,

$$\int d^3x\rho(\mathbf{x}, t) = \langle\psi, \psi\rangle_M \tag{7.5}$$

When $G = \pm i$ and $U = V = I$, (7.3) agrees with the usual result for the Dirac equation,

$$\rho_D = \psi^\dagger\psi \tag{7.6a}$$

$$\mathbf{j}_D = \mp\psi^\dagger\rho_1\boldsymbol{\sigma}\psi \tag{7.6b}$$

Our requirements have forced the Dirac equation to be manifestly covariant, so, as is well known, (7.6) gives the components of a four-vector and, as stressed by Foldy (1956), we can regard ρ_D as a "charge density" and $\mathbf{j}_D$ as a "charge current density."

For real values of G, (7.5) becomes

$$\rho = \tfrac{1}{2}[(V\psi)^\dagger\rho_3(U\psi) + (U\psi)^\dagger\rho_3(V\psi)] \tag{7.7a}$$

$$\begin{aligned}\mathbf{j} = -[i(1+G^2)/4m]\,&[(V\psi)^\dagger(1+\rho_1)(\nabla U\psi) + (U\psi)^\dagger(1+\rho_1)(\nabla V\psi)\\ &- (\nabla U\psi)^\dagger(1+\rho_1)(V\psi) - (\nabla V\psi)^\dagger(1+\rho_1)(U\psi)]\\ &- (G/2)[(V\psi)^\dagger\rho_2\boldsymbol{\sigma}(U\psi) + (U\psi)^\dagger\rho_2\boldsymbol{\sigma}(V\psi)]\end{aligned} \tag{7.7b}$$

For a given choice of the generally nonlocal operators U and V that is consistent with (7.1), it may be possible to argue the interpretation of ρ as a "probability density" and $\mathbf{j}$ as a "probability current density." However, as emphasized by Foldy (1956), if we wish to find a "charge density" ρ and a "charge current density" $\mathbf{j}$, they must transform as the components of a four-vector; in general, ρ and $\mathbf{j}$ defined by (7.7) do not possess this property. For a real value of G, do there exist operators U and V such that ρ and $\mathbf{j}$ yield a satisfactory "charge density" and "charge current density"? At the present time we are unable to answer this question, although it is clear that if the required U and V do exist, they will depend on G and also upon $\theta_+(p)$ and $\theta_-(p)$, the functions appearing in (6.5).

8. *Summary and Discussion*

For arbitrary values of G, the four by four matrix operator H in (2.1) yields the relativistic relation between the squares of the energy, the momentum, and the mass of a free particle, and for $G \neq 0$ it commutes with the angular momentum operator $\mathbf{J} = \mathbf{x} \times \mathbf{p} + \boldsymbol{\sigma}/2$, but not with $\mathbf{x} \times \mathbf{p}$ alone. Thus, for $G \neq 0$ it is natural to interpret H as a Hamiltonian describing spin-$\frac{1}{2}$ particles with both energy signs. However, only when $G = \pm i$ or when G is real does the theory possess a simple metric that commutes with $\mathbf{x}$; in the former case one has the usual Dirac Hamiltonian with a positive definite metric and in the latter case one has a negative normalization for negative energy states. Therefore, in contrast to the Dirac theory, for which the expectation value of the Hamiltonian is indefinite, the expectation value of H in the new theory is positive definite.

If $\mathbf{x}$ is required to transform like a vector under any discrete symmetry operation, then one has invariance under both CPT and charge conjugation. On the other hand, unlike the Dirac theory, which is invariant separately under space inversion and under time inversion, the new theory is not. Although the theory is invariant under proper orthochronous Poincaré transformations, it is not manifestly covariant, and it does not appear possible to find a boost generator that is a local operator.

Our analysis for real values of G "has only provided us with the raw material for the construction of a complete physical theory," to quote Foldy (1956), and whether it actually leads to fundamentally new physical predictions depends on whether one can consistently introduce interactions. In particular, the theory should remain covariant and continue to satisfy

$$H = \rho_3 H^\dagger \rho_3 \tag{8.1a}$$

$$\mathbf{K} = \rho_3 \mathbf{K}^\dagger \rho_3 \tag{8.1b}$$

In the case of coupling to an external electromagnetic field, minimal coupling, although necessary for gauge invariance, does not necessarily yield a covariant theory when one starts from a free-particle theory that is not manifestly covariant; additional terms depending on $\mathbf{E}$ and $\mathbf{B}$ may be required. For example, the Sakata-Taketani Hamiltonian in an external electromagnetic field (Sakata & Taketani, 1940; Heitler, 1943) contains terms with $\mathbf{S} \cdot \mathbf{B}$, where $\mathbf{S}$ is the spin. A significant difference is that the boost operator for the free particle Sakata-Taketani spin-1 Hamiltonian (Guertin, 1974, 1975a) is a local operator, whereas in the present case the introduction of interactions is complicated by the nonlocal nature of the boost generator. In addition, the Sakata-Taketani spin-1 Hamiltonian in an external field can be obtained directly from the corresponding manifestly covariant Duffin-Kemmer-Petiau equation with minimal coupling, whereas we have no manifestly covariant equation corresponding to our new Hamiltonian.

Even if one can introduce electromagnetic interactions in a covariant and gauge-invariant manner, we are reminded by Foldy (1956) that the resulting theories should "give no contradiction to an appropriate causality condition, such as that physically observable effects are not propagated with a velocity greater than the velocity of light, at least on a macroscopic scale." Our equation is not alone in facing this problem; to date no equation has been found that describes the electromagnetic interactions of particles of spin $J \geqslant \frac{3}{2}$ in a causal manner (see, e.g., Velo & Zwanziger, 1969a, b; Wightman, 1971; Velo, 1972; Shamaly & Capri, 1974).

It should, of course, also be possible to second-quantize the theory with real values of G in a consistent manner. Because the normalization in the single-particle theory discussed here is not positive definite and because the energy is, second-quantization would be inconsistent with Fermi statistics. In another paper we shall discuss the possibility of using Bose statistics to second-quantize the free-particle theory.

Our assumption that $\mathbf{x}$ commutes with the metric plays an essential role, since there exists an equivalence relation between the free-particle Dirac Hamiltonian and the Hamiltonian for a real value of G. To distinguish between the various Hamiltonians, let H_D be that for the Dirac theory and H_G that for a given real value of G, and let W_D and W_G be the corresponding Foldy-Wcuthuysen operators introduced in Section 5. Then, according to (5.7c),

$$H_G = W_G W_D^{-1} H_D W_D W_G^{-1} \tag{8.2}$$

However, this transformation carries the positive definite norm

$$\langle \psi_D, \psi_D \rangle_I = \int d^3x \psi_D{}^\dagger(\mathbf{x}, t)\psi_D(\mathbf{x}, t) \tag{8.3a}$$

into the positive definite norm

$$\langle \psi_G, \psi_G \rangle_{\rho_3 H_G} = \int d^3x \psi_G{}^\dagger(\mathbf{x}, t)\rho_3 H_G \psi_G(\mathbf{x}, t) \tag{8.3b}$$

where

$$\psi_G = W_G W_D^{-1} \psi_D \tag{8.4}$$

The possibility of using the positive definite metric $\rho_3 H_G$ with the Hamiltonian for a real value of G was mentioned in Section 3. One can similarly show that under the transformation just introduced the space inversion and time inversion operators for the Dirac theory are transformed into operators that depend on $\mathbf{p}$.

A possible objection to the approach developed here lies in the difficulty of interpreting the significance of $\mathbf{x}$, which is a different operator for each value of G. The operator in the Foldy canonical representation that corresponds to $\mathbf{x}$ in one of our representations, for $G = \pm i$ or G real, is

$$\begin{aligned} \mathbf{x} &= W^{-1}\mathbf{x}W \\ &= \mathbf{x} + \{E[(1+G^2)E + (1-G^2)m]\}^{-1} \left\{ \frac{G^2 m}{E+m} \boldsymbol{\sigma} \times \mathbf{p} \right. \\ &\quad + i\frac{(1+G^2)(E+m)}{2E}\rho_1 \mathbf{p} - iGm\rho_2 \boldsymbol{\sigma} - \frac{G}{2}(1+G^2)(E-m)(\rho_3 + i\rho_2)\boldsymbol{\sigma} \\ &\quad \left. + \frac{G}{E+m}\left[(1+G^2)(\rho_3 + i\rho_2) + i\frac{Gm}{E}\rho_2\right] \boldsymbol{\sigma}\cdot\mathbf{p}\mathbf{p} \right\} \end{aligned} \tag{8.5b}$$

and thus depends on G. In the Dirac theory and in any other theory in which the boost operator has the form $\mathbf{K} = \frac{1}{2}[x, H]_+$, (Fuschich et al., 1971; Kolsrud, 1971; Guertin, 1974, 1975a), **x** is a canonical "position operator," since it has the correct transformation properties under boosts (Currie et al., 1963; Durand, 1973), but **x** for any real value of G in our theory transforms in a complicated manner, since $\mathbf{\Gamma} \neq 0$ in (6.1a). However, one also has $\mathbf{\Gamma} \neq 0$ in the Sakata-Taketani spin-1 boost generator and its arbitrary spin generalization (Guertin, 1974, 1975a), so **x** is also not a canonical "position operator" in such theories.

What is the physical significance of G, if it has any? A possible clue is obtained by introducing minimal coupling into the Hamiltonian (2.1), even though the result is not covariant. One finds that the "magnetic moment" is proportional to G^2, which means that for real values of G it is opposite in direction to that of a Dirac particle with the same charge, i.e., if the charge of the particle is positive the magnetic moment is antiparallel to its spin. This might indicate that our equation describes particles with structure; thus, if the particle has a positive charge, there might be a concentration of positive charge near its center and a thin layer of negative charge near its "surface". Of course, the extra terms that must be introduced into the equation to obtain covariance, if it is indeed possible to do so, might yield the usual type of magnetic moment parallel to the spin for a positively charged particle and antiparallel for a negatively charged one.

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